PMM U.S.S.R., Vol. 46, No. 5, pp. 637-644, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00

UDC 517.946

STATISTICAL EXTREMAL PROBLEMS AND UNIQUE SOLVABILITY OF THE THREE-DIMENSIONAL NAVIER-STOKES SYSTEM UNDER ALMOST ALL INITIAL CONDITIONS*

A.V. FURSIKOV

A probabilistic measure μ_0 is constructed on the functional space of initial conditions of the mixed boundary problem, for the three-dimensional Navier-Stokes (N-S) system. The measure is such, that for μ_0, i.e., for almost all initial conditions, the boundary value problem has a unique solution. To construct μ_0 it is found necessary to solve a certain extremal problem. Theorems of existence and uniqueness of the solution to this extremal problem are proved.

1. Introduction. We consider, in the cylinder $Q = [0, T] \times \Omega$ where $T > 0, \Omega \subset \mathbf{R}^3$ is a bounded region with boundary $\partial\Omega \in C^\infty$ the following N-S system:

$$\dot{\mathbf{y}}(t, \mathbf{x}) - \nu\Delta\mathbf{y} + (\mathbf{y}, \nabla)\mathbf{y} = -\nabla p + \mathbf{f}, \ \operatorname{div}\mathbf{y} = 0 \tag{1.1}$$

Here $t \in (0, T)$, $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$, $\mathbf{y} = (y_1, y_2, y_3)$ is the velocity, $p(t, \mathbf{x})$ is the pressure, $\nu > 0$ is viscosity, $\mathbf{f} = (f_1, f_2, f_3)$ is the external force and $\mathbf{y}^{\cdot} = \partial\mathbf{y}/\partial t$. The following adhesion condition is specified on the side surface $(0, T) \times \partial\Omega$ of the cylinder:

$$(t, \mathbf{x}) \in (0, T) \times \partial\Omega, \ \mathbf{y} = 0 \tag{1.2}$$

and the initial condition has the form

$$t = 0, \ \gamma_0\mathbf{y} = \mathbf{y}_0 \tag{1.3}$$

where γ_0 is the contraction operator of the function $\mathbf{y}(t, \mathbf{x})$ at $t = 0: \gamma_0\mathbf{y} = \mathbf{y}(0, \mathbf{x})$. The theorem on unique solvability of the problem (1.1)-(1.3) at any $(\mathbf{f}, \mathbf{y}_0)$ belonging to the corresponding functional spaces, is not proved. It is only shown that for any $\mathbf{y}_0$ the problem (1.1)-(1.3) has a unique solution provided that $\mathbf{f}$ belongs to some set $F(\mathbf{y}_0)$ dense in the space of the right-hand sides /1/.

Let $\mathbf{f}$ denote an arbitrary fixed external force independent of time t. Below we give a method of defining such a probabilistic measure μ_0 on the space of initial conditions $\{\mathbf{y}_0\}$, that for μ_0, i.e., almost all $\mathbf{y}_0$, the problem (1.1)-(1.3) has a unique solution.

The method is based on solving a certain statistical extremal problem related to the N-S system. The extremal problem in question is a statistical analog of the control problems which have, at certain initial values, more than one solution /2/. Nevertheless, as we show below, this statistical extremal problem has a unique solution. The measure μ_0 is found uniquely from the solution of the statistical extremal problem. However, the extremal problem itself contains parameters and this implies the existence of many measures μ_0 which can be obtained using the proposed method. We shall now define the notation and remind of certain concepts necessary for formulating and solving the problem.

2. Functional spaces. We denote by $\|\cdot\mid X\|$ the norm of the Banach space X. We recall that the set of functions $u(\mathbf{x}), \mathbf{x} \in \Omega$ with finite norm

$$\|u \mid W_2^k(\Omega)\| = \Big(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha u(\mathbf{x})|^2 d\mathbf{x}\Big)^{1/2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiple index and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D^\alpha u = \partial^{|\alpha|}u/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}$, is called the Sobolev space $W^k_2(\Omega)$. Let

$$V = \{\mathbf{v}(\mathbf{x}) \in (C_0^\infty(\Omega))^3 : \operatorname{div}\mathbf{v} = 0\}$$

and H^k be the closure of V in $(W_2^k(\Omega))^3$ for $k = 0, 1$, $H^2 = H^1 \cap (W_2^2(\Omega))^3$ where $(W_2^k(\Omega))^3$ is a Sobolev space of three-dimensional vector fields. The norm $\|\cdot\|_k$ of the space H^k is given by the relation

*Prikl.Matem.Mekhan.,46,No.5.pp.797-805,1982

$$\| \mathbf{u} \|_k = \| \mathbf{u} \,|\, (W_2{}^k(\Omega))^3 \|, \quad k = 0, 1, 2$$

If X is a Banach space, then $L_p(0, T; X)$ denotes a set of functions defined on $(0, T)$ and meaningful in X, for which the norm

$$\| u \,|\, L_p(0, T; X) \| = \Big(\int_0^T \| u(t) \,|\, X \|^p \, dt \Big)^{1/p}$$

is finite. Below we use the following notation:

$$L_p{}^s = L_p(0, T; H^s), \quad \| \mathbf{u} \,|\, L_p{}^s \| = \| \mathbf{u} \,|\, L_p(0, T; H^s) \|$$

Let us set

$$H^{1,2} = \{ \mathbf{y}(t) \in L_2{}^2 : \mathbf{y}^{\cdot} \in L_2{}^{\circ} \}, \quad \| \mathbf{y} \,|\, H^{1,2} \|^2 = \| \mathbf{y} \,|\, L_2{}^2 \|^2 + \| \mathbf{y}^{\cdot} \,|\, L_2{}^{\circ} \|^2$$

In what follows, we use the tensor products of the space H° and tensor products of the vectors $\mathbf{y} \in H^{\circ}$ (the corresponding definitions are given in /3/). We also use the notation

$$\otimes^k H^{\circ} = H^{\circ} \otimes \ldots \otimes H^{\circ} \ (k \ \text{times}), \quad \otimes^k \mathbf{y} = \mathbf{y} \otimes \ldots \otimes \mathbf{y} \ (k \ \text{times})$$

The norm of the space $\otimes^k H^{\circ}$ is denoted by $\| \cdot \|_{(k)}$ and the scalar product by $\langle \cdot, \cdot \rangle_{(k)}$.

3. Probabilistic measures and their moments.

We denote by $B(X)$ the σ-algebra of the Borel subsets of the Banach space X. Let $\mu(dy_0)$ be the probabilistic measure defined on $B(H^{\circ})$ and satisfying the condition

$$\int \| \mathbf{y}_0 \|_0{}^k \mu(dy_0) < \infty \quad \forall k > 0 \tag{3.1}$$

Here and henceforth the integral will be taken, unless indicated otherwise, over the whole domain of definition of the measure, which in the present case is H°.

We define on the Hilbert space $\otimes^k H^{\circ}$ the functional

$$F_k(\varphi) = \int \langle \otimes^k \mathbf{y}_0, \varphi \rangle_{(k)} \mu(dy_0) \tag{3.2}$$

Since by virtue of the Cauchy–Buniakowski inequality

$$| F_k(\varphi) | \leqslant \int \| \mathbf{y}_0 \|_0{}^k \mu(dy_0) \| \varphi \|_{(k)} \tag{3.3}$$

it follows from (3.1) that the functional (3.2) is continuous. Therefore by virtue of the Riesz theorem there exists a vector $m_k \in \otimes^k H^{\circ}$ such that

$$\langle m_k, \varphi \rangle_{(k)} = \int \langle \otimes^k \mathbf{y}_0, \varphi \rangle_{(k)} \mu(dy_0) \ \forall \varphi \in \otimes^k H^{\circ} \tag{3.4}$$

The element m_k is called the k-th moment of the measure μ.

In addition to the measure defined on the space H° of vector fields depending on $\mathbf{x} \in \Omega$, we also consider the measures on the space of vector fields depending on $t \in (0, T)$, $\mathbf{x} \in \Omega$. Let P be a probabilistic measure on $B(H^{1,2})$. We define the measure $\gamma_0{}^*P$ using the relations

$$\gamma_0{}^*P(\omega_0) = P(\gamma_0^{-1}\omega_0) \quad \forall \omega_0 \in B(H^{\circ}) \tag{3.5}$$

$$\gamma_0 : H^{1,2} \to H^1 \subset H^{\circ} \tag{3.6}$$

Here γ_0 is the contraction operator of $\mathbf{y}(t, \mathbf{x})$ at $t = 0$ and $\gamma_0{}^{-1}\omega_0$ is the complete inverse image of the set ω_0 under the mapping (3.6). The mapping (3.6) is continuous /4/, therefore the formula (3.5) determines the probabilistic measure on $B(H^{\circ})$ concentrated on H^1. The measure $\gamma_0{}^*P$ is a contraction of the measure P at $t = 0$. If

$$\int \| \mathbf{y} \,|\, H^{1,2} \|^k P(dy) < \infty \quad \forall k > 0$$

then by virtue of (3.5) and (3.6)

$$\int \| \mathbf{y}_0 \|_0{}^k \gamma_0{}^*P(dy_0) = \int \| \gamma_0 \mathbf{y} \|_0{}^k P(dy) < c^k \int \| \mathbf{y} \,|\, H^{1,2} \|^k P(dy) < \infty$$

and this yields the moments M_k of the measure γ_0^*P

$$\langle M_k, \varphi \rangle_{(k)} = \int \langle \otimes^k \mathbf{y}_0, \varphi \rangle_{(k)} \gamma_0{}^*P(dy_0) = \int \langle \otimes^k \gamma_0 \mathbf{y}, \varphi \rangle_{(k)} P(dy) \tag{3.7}$$

4. Statistical solutions. We denote by $\langle\cdot,\cdot\rangle$ the scalar product in the spaces $(L_2(\Omega))^3$ and H°, and by $[\cdot,\cdot]$ the scalar product in L_2° and $(L_2(Q))^3$:

$$\langle \mathbf{u}, \mathbf{v}\rangle = \int_\Omega \sum_{j=1}^{3} u_j(x) v_j(x)\, dx, \quad [\mathbf{u}, \mathbf{v}] = \int_0^T \langle \mathbf{u}(t,\cdot), \mathbf{v}(t,\cdot)\rangle\, dt$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$. Since for any $\mathbf{v} \in H^\circ$, $p \in W_2^1(\Omega)$ $\langle \nabla p, \mathbf{v}\rangle = 0$, therefore scalar multiplying in $(L_2(Q))^3$ both parts of the first equation of (1.1) by $\mathbf{z} \in L_2^0$, we obtain

$$[\mathbf{y}^{\cdot} - \nu\Delta\mathbf{y} + (\mathbf{y},\nabla)\mathbf{y} - f, \mathbf{z}] = 0 \quad \forall \mathbf{z} \in L_2^\circ \tag{4.1}$$

From the theorem on orthogonal expansion of $(L_2(\Omega))^3$ into the subspaces of solenoidal and potential vector fields /5/ it follows that $y \in H^{1,2}$ satisfies the relation (4.1) if and only if there exists $p \in L_2(0,T; W_2^1(\Omega))$ such that the pair $(\mathbf{y}, p)$ satisfies (1.1). Therefore in what follows we shall use (4.1) instead of (1.1). We shall call the function $\mathbf{y} \in H^{1,2}$ satisfying (4.1), an individual solution of the N-S system. We denote by Θ_+ the set of probabilistic measures defined on $B(H^{1,2})$. If X is a Banach space, then $C_b(X)$ is a space of continuous bounded functions on X. The measure $P \in \Theta_+$ is called a statistical solution of the N-S system, provided that for any $\mathbf{z} \in L_2^\circ$ and $\varphi \in C_b(H^{1,2})$ the functional $\mathbf{y} \to [\mathbf{y}^{\cdot} - \nu\Delta\mathbf{y} + (\mathbf{y},\nabla)\mathbf{y} - \mathbf{f}, \mathbf{z}]$ is P-integrable and

$$\int [\mathbf{y}^{\cdot} - \nu\Delta\mathbf{y} + (\mathbf{y},\nabla)\mathbf{y} - \mathbf{f}, \mathbf{z}]\, \varphi(y)\, P(dy) = 0 \tag{4.2}$$

We note that for every individual solution we can easily find the corresponding statistical solution. Indeed, let χ be the individual solution and let us consider the measure δ_χ defined by the relation

$$\forall \omega \in B(H^{1,2}) \quad \delta_\chi(\omega) = \begin{cases} 1, & \text{if} \quad \chi \in \omega \\ 0, & \text{if} \quad \chi \notin \omega \end{cases} \tag{4.3}$$

Then by virtue of (4.2) and (4.3) we have for any $z \in L_2^\circ$, $\varphi \in C_b(H^{1,2})$

$$\int [\mathbf{y}^{\cdot} - \nu\Delta\mathbf{y} + (\mathbf{y},\nabla)\mathbf{y} - \mathbf{f}, \mathbf{z}]\, \varphi(\mathbf{y})\, \delta_\chi(dy) = [\chi^{\cdot} - \nu\Delta\chi + (\chi,\nabla)\chi - \mathbf{f}, \mathbf{z}]\, \varphi(\chi) = 0$$

and δ_χ is therefore a statistical solution.

5. Formulation of the extremal problem. Let a probabilistic measure $\mu(dy_0)$ be given on the space of initial conditions H°, the measure satisfying the condition

$$\int \exp \|y_0\|_0^2\, \mu(dy_0) < \infty \tag{5.1}$$

We denote by m_k the moments of the measure μ given by (3.4). Consider the functional

$$J(P) = \int \exp \|\mathbf{y} \,|\, L_2^2\|^2 P(dy) + N \sum_{k=1}^{\infty} \frac{1}{k!} \|m_k - M_k\|_{(k)}^2 \tag{5.2}$$

where M_k are the moments of the measure $\gamma_0^* P$ determined by equality (3.7), and $N > 0$ is a parameter. We consider the following extremal problem:

$$J(P) \to \inf \tag{5.3}$$

$$P \in \Theta_+, \ \int [\mathbf{y}^{\cdot} - \nu\Delta\mathbf{y} + (\mathbf{y},\nabla)\mathbf{y} - \mathbf{f}, z]\, \varphi(\mathbf{y})\, P(dy) = 0, \quad \forall \mathbf{z} \in L_2^\circ, \quad \varphi \in C_b(H^{1,2}) \tag{5.4}$$

where $J(P)$ is the functional (5.2). Thus, out of all statistical solutions, we have to choose a solution for which the integral

$$\int \exp \|\mathbf{y} \,|\, L_2^2\|^2 P(dy) \tag{5.5}$$

is not very large and the contraction at $t = 0$ has moments as close as possible to the given moments m_k. We call the admissible element the statistical solution on which the functional (5.2) is finite. Let us denote by A the set of admissible elements, and let $A \neq \varnothing$. Then the measure $P' \in A$ such that

$$J(P') = \inf_{P \in A} J(P)$$

will be a solution of the problem (5.3), (5.4).

6. Existence of the solution to the problem (5.3), (5.4). Lemma 6.1. Let $\mathbf{f} \in H^\circ$ and $\mathbf{y}$ be an individual solution. Then

$$\| \gamma_0 \mathbf{y} \|_0 \leqslant c(1 + \| \mathbf{y} \mid L_2{}^2 \|) \tag{6.1}$$

$$\| \mathbf{y} \mid L_\infty{}^\circ \| \leqslant c(1 + \| \mathbf{y} \mid L_2{}^2 \|) \tag{6.2}$$

$$\| \mathbf{y}^\cdot \mid L_2{}^\circ \|^2 \leqslant c(1 + \| \mathbf{y} \mid L_2{}^2 \|^3) \tag{6.3}$$

where c depends on f.

Proof. Assuming that $\mathbf{z} = r(t)\mathbf{v}$ in (4.1), where $r(t) \in L_2(0, T)$, $\mathbf{v} \in H^\circ$, we find that

$$\langle \mathbf{y}^\cdot - \nu \Delta \mathbf{y} + (\mathbf{y}, \nabla)\mathbf{y} - \mathbf{f}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in H^\circ \tag{6.4}$$

for almost all $t \in [0, T]$. Setting in (6.4) $\mathbf{v} = \mathbf{y}(t, \cdot)$, we obtain the relation

$$\frac{1}{2} \frac{d}{dt} \| \mathbf{y} \|_0^2 + \nu \| \nabla \mathbf{y} \|_0^2 - \langle \mathbf{f}, \mathbf{y} \rangle = 0 \tag{6.5}$$

Let $\varphi(t) = (T - t)/T$. Multiplying (6.5) by $\varphi(t)$, adding $\varphi^\cdot \| \mathbf{y} \|_0^2/2$ to both sides of the resulting expression and integrating with respect to t, we obtain

$$\frac{1}{2} \| \gamma_0 \mathbf{y} \|_0^2 = \int_0^T \left(\nu\varphi \| \nabla \mathbf{y} \|_0^2 - \varphi \langle \mathbf{f}, \mathbf{y} \rangle - \frac{1}{2} \varphi^\cdot \| \mathbf{y} \|_0^2 \right) dt \leqslant c(1 + \| \mathbf{y} \mid L_2{}^1 \|^2)$$

from which, by virtue of the continuity of the inclusion $H^2 \subset H^1$, follows (6.1). The relation (6.2) follows from (6.1) and the energetic inequality of the N-S system. In proving (6.3) we use the norms $\| \cdot \|_s$ of the spaces H^s with real s, which are defined in e.g. /1/.

Let us find the operator $b(\mathbf{u}, \mathbf{v})$ using the relation $\langle b(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = \langle (\mathbf{u}, \nabla)\mathbf{v}, \mathbf{w} \rangle$. In /6/ it is shown that

$$\| b(\mathbf{u}, \mathbf{v}) \|_{-\gamma} \leqslant c \| \mathbf{u} \|_\alpha \| \mathbf{v} \|_{\beta+1} \tag{6.6}$$

provided that $\alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0$ and $\alpha + \beta + \gamma > {}^3/_2$. From (6.6) it follows that

$$\| b(\mathbf{y}, \mathbf{y}) \mid L_2^{-3/2} \| \leqslant c \| \mathbf{y} \mid L_\infty^0 \| \cdot \| \mathbf{y} \mid L_2{}^2 \|$$

therefore from (6.4) and (6.2) we obtain the inequality

$$\| \mathbf{y}^\cdot \mid L_2^{-3/2} \| \leqslant c(1 + \| \mathbf{y} \mid L_2{}^2 \|^2) \tag{6.7}$$

From this inequality we find, as in /3/, that

$$\| \mathbf{y} \mid L_\infty^{1/2} \|^2 \leqslant c \| \mathbf{y}^\cdot \mid L_2^{-1/2} \| \cdot \| \mathbf{y} \mid L_2{}^2 \| \leqslant c(1 + \| \mathbf{y} \mid L_2{}^2 \|^3) \tag{6.8}$$

and by virtue of (6.6) and (6.8) we have

$$\| b(\mathbf{y}, \mathbf{y}) \mid L_2{}^\circ \| \leqslant c \| \mathbf{y} \mid L_\infty^{1/2} \| \, \| \mathbf{y} \mid L_2{}^2 \| \leqslant c_1(1 + \| \mathbf{y} \mid L_2{}^2 \|^{3/2})$$

from which, together with (6.4), we obtain (6.3).

Lemma 6.2. Let $\mathbf{f} \in H^\circ$. Then the set A of admissible elements is nonempty.

Proof. We know /5/ that for any $\mathbf{f} \in H^\circ$ there exists a solution $\boldsymbol{\chi} \in H^2$ of the stationary problem

$$\langle -\nu\Delta\boldsymbol{\chi} + (\boldsymbol{\chi}, \nabla)\boldsymbol{\chi} - \mathbf{f}, \boldsymbol{\psi} \rangle = 0 \quad \forall \boldsymbol{\psi} \in H^\circ$$

Clearly, $\boldsymbol{\chi}$ is an individual solution of the nonstationary problem (4.1), therefore δ_χ is a statistical solution. We shall show that $J(\delta_\chi) < \infty$. Indeed,

$$\int \exp \| \mathbf{y} \mid L_2{}^2 \|^2 \delta_\chi(dy) = \exp \| \chi \mid L_2{}^2 \|^2 = \exp T \| \chi \|_2^2 < \infty \tag{6.9}$$

By virtue of (3.5), (4.3) and (3.7), the moments M_k of the measure $\gamma_0{}^*\delta_\chi$ can be found using the equation

$$M_k = \int \otimes^k \gamma_0 \mathbf{y} \delta_\chi(dy) = \otimes^k \chi$$

therefore we have

$$\| M_k \|_{(k)}^2 = \| \chi_0 \|_0^{2k} \tag{6.10}$$

From (3.2) – (3.4) it follows that

$$\| m_k \|_{(k)} \leqslant \int \| \mathbf{y}_0 \|_0^k \mu(dy_0)$$

therefore by virtue of (5.1) we have

$$\sum_{k=1}^\infty \frac{\| m_k \|_{(k)}^2}{k!} \leqslant \int \exp \| \mathbf{y}_0 \|_0^2 \mu(dy_0) < \infty \tag{6.11}$$

Now from (6.9) – (6.11) it follows that $J(\delta_\chi) < \infty$, and hence $A \neq \varnothing$.

Lemma 6.3. Let P be a statistical solution. Then a set $W \in B(H^{1,2})$ exists such that $P(W) = 1$ and W consists of individual solutions of the N-S system. The lemma is proved in the same manner as the analogous assertion in /3/.

Lemma 6.4. Let $P_i \in A$ be a sequence of measures satisfying the inequality

$$J(P_i) \leqslant \lambda \tag{6.12}$$

where λ is independent of i. Then a subsequence P_j of the sequence P_i exists, converging weakly on $C(0, T; H^\circ)$ to the measure $P \in \Theta_+$, the latter representing a statistical solution.

Proof. Let W_i be a set of P_i with full measure, consisting of the individual solutions. The inequality (6.3) holds for any $\mathbf{y} \in W_i$. Raising both sides of this inequality to the power $k/2$ and integrating with respect to P_i we find, by virtue of (6.12), that

$$\int \| \mathbf{y} \mid H^{1,2} \|^k P_i(dy) \leqslant c_k \tag{6.13}$$

where c_k is independent of i. Since the inclusion $H^{1,2} \subset C(0, T; H^\circ)$ is completely continuous /3/, then from (6.13) and Iu.V. Prokhorov theorem /7,3/ it follows that a subsequence P_j exists, converging weakly on $C(0, T; H^\circ)$ to some measure P. Obviously, P is a probabilistic measure on $C(0, T; H^\circ)$.

Let $\{e_j\}$ be an orthonormed basis in $H^{1,2}$, and let for any j $e_j \in C^1(0, T; H^4)$ (definition of the space H^4 is given in /1/). We denote by G_r the orthoprojector in $H^{1,2}$ onto the subspace $[e_1, \ldots, e_r]$ generated by the first r vectors e_j. The functional $\mathbf{y} \to \| G_r \mathbf{y} \mid H^{1,2} \|^k$, is defined on $H^{1,2}$ and is extended to $C(0, T; H^\circ)$ by virtue of the continuity. By virtue of (6.13) we have

$$\int \| G_r \mathbf{y} \mid H^{1,2} \|^k P_j(dy) \leqslant c_k \tag{6.14}$$

Passing in (6.14) to the limit as $j \to \infty$, with help of the lemma 3.2 given in /3/, and using the Beppo-Levi theorem we find, as $r \to \infty$,

$$\int \| \mathbf{y} \mid H^{1,2} \|^k P(dy) \leqslant c_k \tag{6.15}$$

From this it follows that the measure P is concentrated on $H^{1,2}$, and hence $P \in \Theta_+$.

Integrating the expression $[\mathbf{y}' - \nu\Delta\mathbf{y} + (\mathbf{y}, \nabla)\mathbf{y} - \mathbf{f}, \mathbf{z}]$ by parts we find, that for any $\mathbf{z} \in C^1(0, T; H^2)$ the functional $\mathbf{y} \to [\mathbf{y}' - \nu\Delta\mathbf{y} + (\mathbf{y}, \nabla)\mathbf{y} - \mathbf{f}, \mathbf{z}]$ is continuous on $C(0, T; H^\circ)$. Therefore, if $\varphi \in C_b(C(0, T; H^\circ))$ vanishes outside some bounded set, then the integrand expression in (4.2) belongs to $C_b(C(0, T; H^\circ))$. Since P_i is a statistical solution, we find by substituting P_i into (4.2) and passing to the limit as $i \to \infty$, that P also satisfies (4.2). Aproximating now $\mathbf{z} \in L_2^\circ$ by the functions $\mathbf{z}_j \in C^1(0, T; H^2)$ and $\varphi \in C_b(H^{1,2})$ by the functions $\varphi_j \in C_b(C(0, T; H^\circ))$ with bounded carriers, we find from the Lebesque theorem and (6.15) that (4.2) holds for any $\mathbf{z} \in L_2^\circ$, $\varphi \in C_b(H^{1,2})$, consequently P is a statistical solution.

Lemma 6.5. If $P_j \in A$ and $P_j \to P$ weakly on $C(0, T; H^\circ)$, then from (6.12) it follows that

$$J(P) \leqslant \lambda$$

Proof. We shall show that for any $\varphi \in \otimes^k H^\circ$

$$\lim_{j\to\infty} \int \langle \otimes^k \mathbf{y}_0, \varphi \rangle_{(k)} \gamma_0^* P_j(dy_0) = \int \langle \otimes^k \mathbf{y}_0, \varphi \rangle_{(k)} \gamma_0^* P(dy_0) \tag{6.16}$$

Let

$$B_r = \{y \in C(0, T; H^\circ) : \| \mathbf{y} \mid C(0, T; H^\circ) \| \leqslant r\}, \quad S_r = \partial B_r$$

i.e. S_r is a boundary of the sphere B_r. Clearly, for any $r > 0$ and $B_r S_r$ are P-measurable and there exists a sequence with $r \to \infty$ such that $P(S_r) = 0 \; \forall r$. Since the functional $\mathbf{y} \to \langle \otimes^k \gamma_0 \mathbf{y}, \varphi \rangle_{(k)}$ is continuous on $C(0, T; H^\circ)$, we have /7/

$$\int_{B_r} \langle \otimes^k \gamma_0 \mathbf{y}, \varphi \rangle_{(k)} P_j(dy) \underset{j\to\infty}{\longrightarrow} \int_{B_r} \langle \otimes^k \gamma_0 \mathbf{y}, \varphi \rangle_{(k)} P(dy) \tag{6.17}$$

Let $\theta_r = C(0, T; H^\circ) \setminus B_r$. By virtue of the Chebyshev inequality, (6.2) and (6.12),

$$\left| \int_{\theta_r} \langle \otimes^k \gamma_0 \mathbf{y}, \varphi \rangle_{(k)} P_i(dy) \right| \leqslant c \int_{\theta_r} \| \mathbf{y} \mid L_\infty^0 \|^k P_j(dy) \leqslant \frac{c'}{r} \int (1 + \| \mathbf{y} \mid L_2^{\circ} \|)^{k+1} P_j(dy) \leqslant c_k / r \tag{6.18}$$

where c_k is independent of r and j. From (3.7), (6.17) and (6.18) follows (6.16). Let M_{kj} be the k-th moment of the measure $\gamma_0{}^*P_j$ while M_k the k-th moment of the measure $\gamma_0{}^*P$. Then (6.16) means that

$$M_{kj} \to M_k \quad \text{as} \quad j \to \infty \quad \text{weakly in} \quad \otimes^k H^\circ \tag{6.19}$$

From (5.2) and (6.12) it follows that for any natural l

$$\int \exp \| y \mid L_2{}^2 \|^2 P_j(dy) + N \sum_{k=1}^{l} \frac{1}{k!} \| m_k - M_{kj} \|_k{}^2 \leqslant \lambda$$

Let us pass in this inequality to the limit as $j \to \infty$, using (6.19) for the second term of the left-hand side, and the arguments used in the course of proving (6.15), for the first term. As a result we find that for any l

$$\int \exp \| y \mid L_2{}^2 \|^2 P(dy) + N \sum_{k=1}^{l} \frac{1}{k!} \| m_k - M_k \|_k{}^2 \leqslant \lambda$$

Passing now to the limit as $l \to \infty$, we obtain the affirmation of lemma. Next we consider the question of existence of a solution of the problem (5.3), (5.4).

Theorem 6.1. Let $f \in H^\circ$. Then the problem (5.3), (5.4) has a solution P'.

Proof. By virtue of Lemma 6.2 $A \neq \varnothing$ and a sequence $P_i \in A$ exists such that

$$\lim_{i \to \infty} J(P_i) = \inf_{P \in A} J(P) \tag{6.20}$$

Clearly, $J(P_i) \leqslant c$ where c is independent of i, therefore according to Lemma 6.4 there exists a subsequence P_j converging weakly to the statistical solution P'. By virtue of Lemma 6.5 $P' \in A$ and

$$J(P') \leqslant \lim_{j \to \infty} J(P_j) \tag{6.21}$$

and from (6.20), (6.21) it follows that P' is a solution of the problem (5.3), (5.4).

7. Unique solvability of the N-S system for almost all initial conditions.

Let P' be a solution of the system (5.3), (5.4). By virtue of Lemma 6.3 there exists a set $W \in B(H^{1,2})$ of P'-complete measure consisting of the individual solutions. Let us put

$$V = \gamma_0 W = \{ y_0 \in H^1 : y_0 = \gamma_0 y \quad \text{for some} \quad y \in W \} \tag{7.1}$$

Then /3/ $\gamma_0{}^*P'(V) = 1$. The definition (7.1) of the set V implies directly that for any $y_0 \in V$ there exists a solution $y \in H^{1,2}$ of the problem (4.1), (1.3). Since not more than one solution of the problem (4.1), (1.3) exists in the space $H^{1,2}$ /3/, this proves the following theorem.

Theorem 7.1. Let P' be a solution of the problem (5.3), (5.4) and $\mu_0 = \gamma_0{}^*P'$ Then for μ_0, i.e. for almost all initial conditions y_0, the problem (4.1), (1.3) has a unique solution $y \in H^{1,2}$

8. Uniqueness of the solution of the problem (5.3), (5.4).

Theorem 8.1. The problem (5.3), (5.4) has not more than one solution.

Proof. Let P_1 and P_2 be two solutions of the problem (5.3), (5.4); $\mu_i = \gamma_0{}^*P_i$, M_{ki} is the k-th moment of the measure μ_i and $i = 1, 2$. By virtue of the strong convexity of any Hilbert norm $\| \cdot \|$ we have

$$\| (h_1 + h_2)/2 \|^2 \leqslant (\| h_1 \|^2 + \| h_2 \|^2)/2$$

and the equality holds when and only when $h_1 = h_2$. Therefore

$$J((P_1 + P_2)/2) \leqslant (J(P_1) + J(P_2))/2 \tag{8.1}$$

and the equality is reached only and only in the case when

$$M_{k,1} = M_{k,2} \tag{8.2}$$

for any k. Since P_1 and P_2 are solutions of problem (5.3), (5.4) while $(P_1 + P_2)/2 \in A$, therefore the equality is attained in (8.1) and this implies that (8.2) holds.

By virtue of (6.1) we have, for sufficiently small $\beta > 0$,

$$\int \exp(\beta \| y_0 \|_0{}^2) \mu_i(dy_0) \leqslant c \int \exp \| y \mid L_2{}^2 \|^2 P_i(dy) < \infty \tag{8.3}$$

Let

$$\varkappa_{k,i} = \int \| y_0 \|_0^k \mu_i (dy_0)$$

The relation (8.3) implies the following inequality:

$$\sum_{k=1}^{\infty} \frac{\beta^k}{k!} \varkappa_{2k,i} < \infty$$

from which, making use of the Stirling formula we obtain

$$\sum_{k=1}^{\infty} (\varkappa_{2k,i})^{-1/(2k)} = \infty \tag{8.4}$$

Using (8.2), (8.4) we derive the relation $\mu_1 = \mu_2$. Let $\{e_j\}$ be an orthonormed basis in H° and G_r an orthoprojector in H° onto the subspace $[e_1, \ldots, e_r]$ generated by the vectors $e_1, \ldots, e_r$. Consider the finite-dimensional projections of the measures μ_i

$$\mu_i^r(\omega) = \mu_i (G_r^{-1}\omega) \quad \forall \omega \in B([e_1, \ldots, e_r])$$

Clearly, the k-th moment $M_{k,i}^r$ of the measure μ_i^r is equal to $(\otimes^k G_r) M_{ki}$ where $\otimes^k G_r$ is the k-th tensor degree of the operator G_r. Therefore by virtue of (8.2), we have

$$M_{k,1}^r = M_{k,2}^r \quad \forall k, r \tag{8.5}$$

Let us set

$$s_{k,i}^r = \int \sum_{j=1}^{r} x_j^k \mu_i^r (dy_0), \quad x_j = \langle y_0, e_j \rangle$$

Clearly, that

$$s_{2k,i}^r \leqslant \int \Big(\sum_{j=1}^{r} x_j^2 \Big)^k \mu_i^r (dy_0) = \int \| G_r y_0 \|_0^{2k} \mu_i (dy_0) \leqslant \varkappa_{2k,i}$$

and hence by virtue of (8.4),

$$\sum_{k=1}^{\infty} (s_{2k,i}^2)^{-1/(2k)} = \infty \quad \forall r \tag{8.6}$$

From (8.6) it follows /8/ that the measure μ_i^r is uniquely determined by its moments $M_{ki}{}^r$, therefore by virtue of (8.5) $\mu_1{}^r = \mu_2{}^r$. Since the latter equality holds for any r, we have $\mu_1 = \mu_2$ from which it follows /3/ that $P_1 = P_2$, which proves the theorem.

As we have already said, the problem (5.3), (5.4) represents a statistical analog of the control problems for a system described by the N-S equations which, as was shown in /2/, have more than one solution under certain initial conditions. Nonuniqueness of the solutions in a determinate case is explained by the fact that in these problems the nonlinearity of the N-S equations results in the nonconvexity of the set of admissible elements. The difference between the determinate and statistical problems lies, roughly speaking, in the fact that in the first case the solution is sought in the class of the δ-measure, and in the second case in a wider class of all probabilistic measures. Under such extension the nonconvex class becomes convex and this leads to the uniqueness of the solution in the case of the statistical problems.

9. Certain variants of the problem (5.3), (5.4). We shall say that the measure μ_0 defined on $B(H^\circ)$ has the property E if for μ_0, almost all y_0, the problem (4.1),(1.3) has a unique solution. We note that the integral (5.5) of the functional (5.2) carries the following functions: 1°. Since the norm $\| y \mid L_2{}^2 \|$ is P-integrable, the measure $\gamma_0 * P$ has the property E since the problem (4.1), (1.3) has at most one solution in the space $L_2{}^2$; 2°. Since $\exp \| y \mid L_2{}^2 \|^2$ is P-integrable, the relation (8.4) can be proved and this ensures that the solution of the problem (5.3), (5.4) is unique. Therefore in constructing the measure μ_0 with the property E we can use, apart from the norm $\| \cdot \mid L_2{}^2 \|$, the norms of the spaces in which the solution of the problem (4.1), (1.3) is unique, e.g., $\| \cdot \mid L_4{}^1 \|$. The functional $\exp \| \cdot \|^2$ can be replaced by the functionals growing less rapidly as $\| y \| \to \infty$. We can, for example, replace the functional (5.2) by the functional

$$J(P) = \int \exp \| y \mid L_4{}^1 \| P(dy) + N \sum_{k=1}^{\infty} \frac{1}{(2k)!} \| m_k - M_k \|_{(k)}^2 \to \inf \tag{9.1}$$

Then the problem (9.1), (5.4) will also have a unique solution P and γ_0*P will have the property E. Reducing the rate of growth in the value of the integrand in (5.5) by considering, e.g., a functional obeying a power law, we find that the infinite series in (5.2) can be conveniently replaced by a finite sum. Let us consider, for example, the problem

$$J(P) = \int \| y \mid L_4^1 \|^4 P(dy) + \| m_1 - M_1 \|_{(1)}^2 + \| m_2 - M_2 \|_{(2)}^2 \to \inf \quad (9.2)$$

for P satisfying the conditions (5.4). The problem (9.2), (5.4) has a solution P while γ_0*P has the property E. An attempt to prove the uniqueness of the solution of (9.2), (5.4) has, however, proved unsuccessfull. It could only be shown that the moments M_1 and M_2 are determined uniquely. The results related to the problem (9.2), (5.4) are given in /9/.

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Translated by L.K.